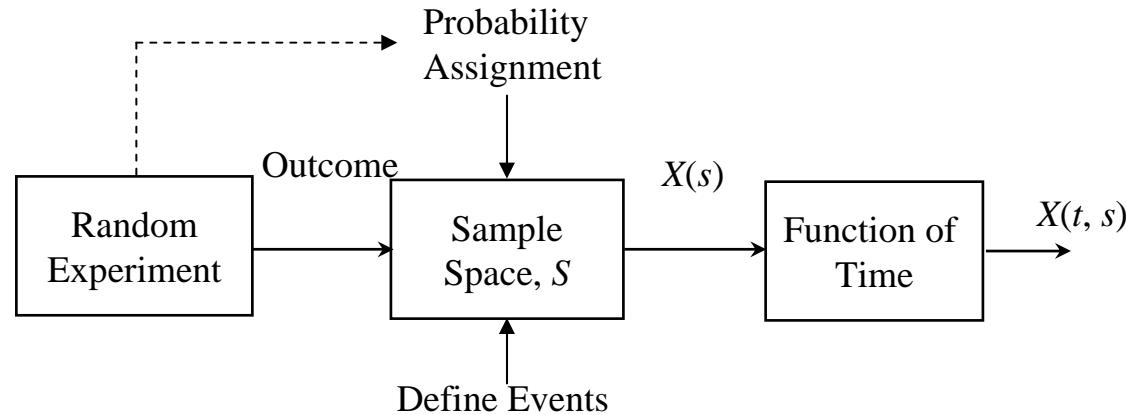
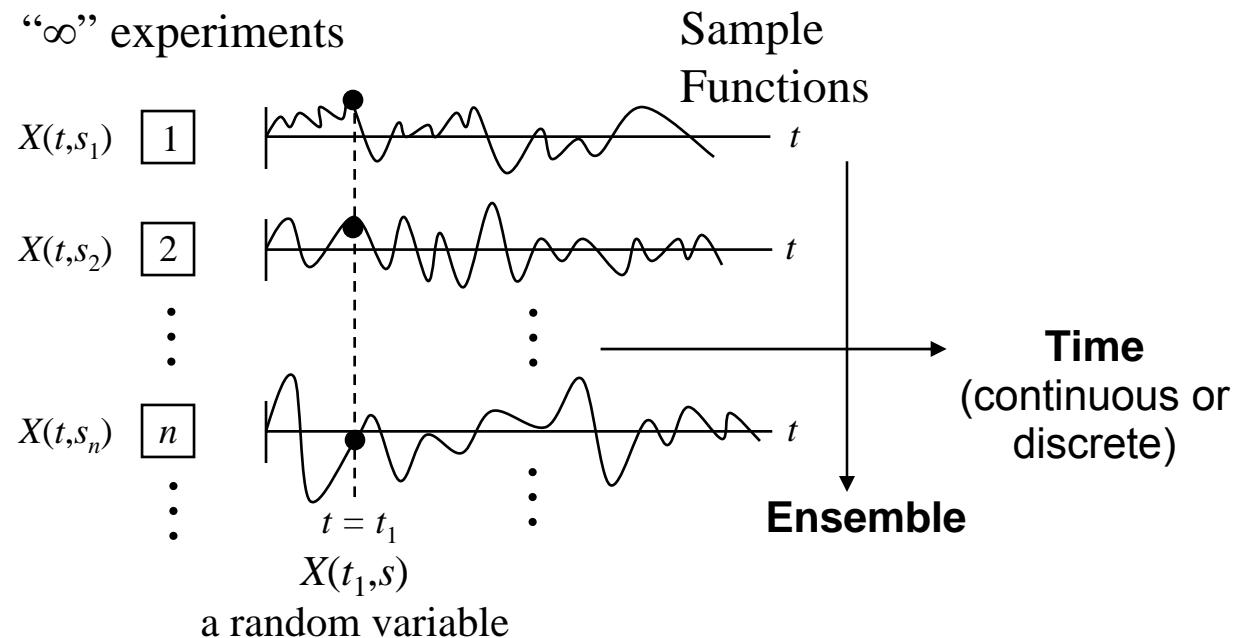


Random Processes

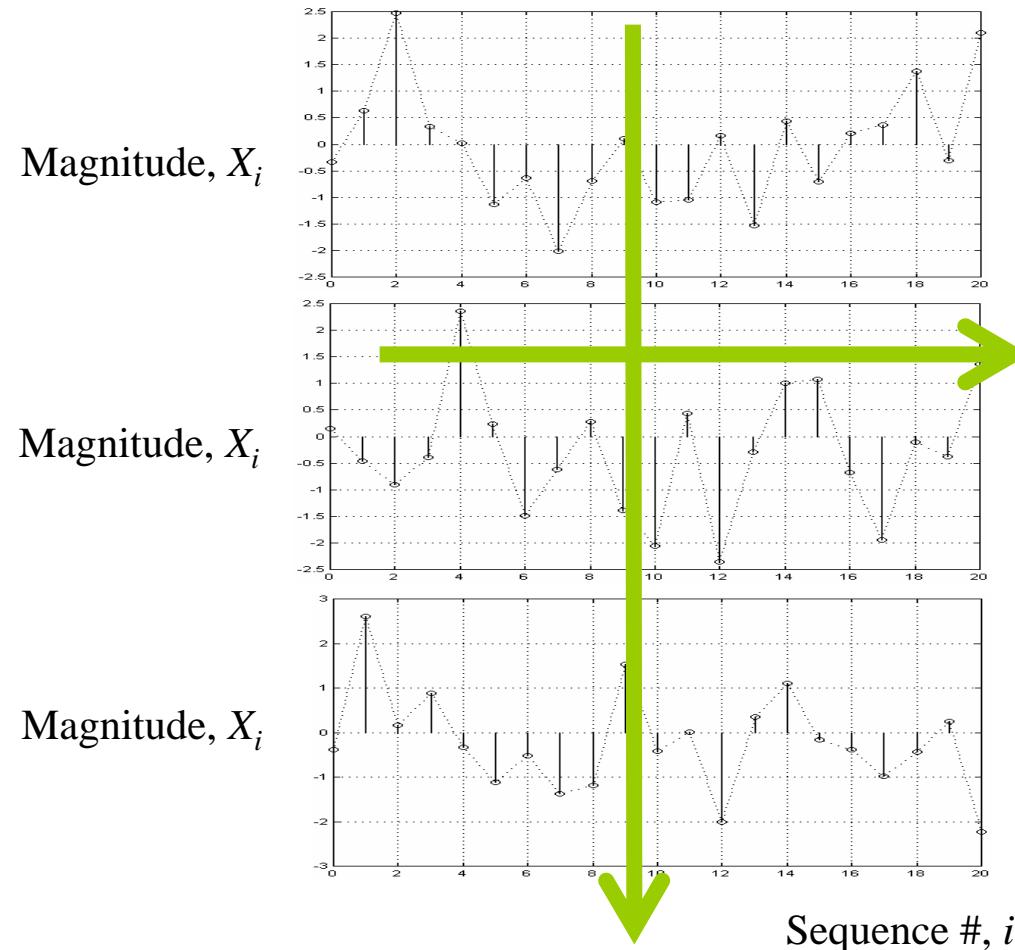


Concept of a random process (Ensemble)



Realizations of a Sequence of Random Variables

$$X(t_i) = 0.5 \cos(\omega t_i) + N(t_i), \quad m_N = 0; \quad \sigma_N^2 = 1;$$



Points Related to Random Processes

- The pdf is associated with a r.v. — along the ensemble.
- The expectations are taken along the ensemble.
- In a physical experiment, there is usually only one sample function available.
- If we can relate “ensemble” to the “time” axis, then the theory of random processes can be applied to analyze the time waveforms or random signals.

Example:

(1) Consider a sample space $S = \{s: -1 \leq s \leq 1\}$

Define a random process



$$X(t, s) = s \cos(2\pi t), \quad -\infty < t < \infty$$

(2) Define a random process



$$X(t, s) = \cos(2\pi t + s), \quad -\pi \leq s \leq \pi$$

Discrete-time form:

Define $X[n, s] = s \cos[2\pi n]$

Define $X[n, s] = \cos[2\pi n + s]$

PDF and CDF of a Random Signal

Each sample, $X(t_i)$, at a time point t_i is a r.v., hence, it has a pdf.

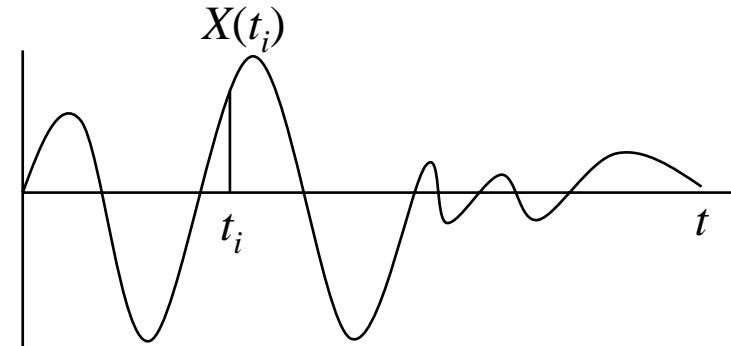
$$X_i = X_i(t_i) \quad \text{is a r.v.} \Rightarrow f_{X_i}(x_i)$$

Two points X_i, X_j have a joint pdf:

$$X_i = X(t_i), \quad X_j = X(t_j) \Rightarrow f_{X_i X_j}(x_i, x_j)$$

We can extend this to n -points characterized by

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}, \text{ a random vector}$$



$$f_{\mathbf{X}}(\mathbf{x}), \quad F_{\mathbf{X}}(\mathbf{x})$$

Moments of a Random Process (*Continuous Time*)

Mean of random signal (first moment)

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x) dx$$

Autocorrelation function of random signal (second moment)

$$R_X(t_1, t_0) = E[X(t_1)X(t_0)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_0 f_{X(t_1)X(t_0)}(x_1, x_0) dx_1 dx_0$$

Autocovariance function (second central moment)

$$\begin{aligned} C_X(t_1, t_0) &= E[(X(t_1) - m_X(t_1))(X(t_0) - m_X(t_0))] \\ &= R_X(t_1, t_0) - m_X(t_1)m_X(t_0) \end{aligned}$$

When $t_1 = t_0$

$$R_X(t_0, t_0) = E[X^2(t_0)] = \int_{-\infty}^{\infty} x_0^2 f_{X(t_0)}(x_0) dx_0$$

$$C_X(t_0, t_0) = E[(X(t_0) - m_X(t_0))^2] = \sigma_X^2(t_0)$$

Moments of a Random Process (*Cont'd.*)

Correlation coefficient

$$\rho_X(t_1, t_0) = \frac{C_X(t_1, t_0)}{\sigma_X^2(t_1)\sigma_X^2(t_0)} = \frac{C_X(t_1, t_0)}{\sqrt{C_X(t_1, t_1)}\sqrt{C_X(t_0, t_0)}}$$

Moments of a Discrete Time Random Process

Similar definitions, except we write

$$m_X[n], \quad R_X[n_1, n_0], \quad \dots \text{etc.}$$

Example: $X(t) = A \cos(\omega t + \phi)$, ϕ is uniform $[-\pi, \pi]$

Find $m_X(t)$ and $R_X(t_1, t_0)$.

$$m_X(t) = E[X(t)] = A \int_{-\pi}^{\pi} \cos(\omega t + \phi) \frac{d\phi}{2\pi} = 0, \text{ independent of time}$$

$$\begin{aligned} R_X(t_1, t_0) &= E[X(t_1)X(t_0)] = \int_{-\pi}^{\pi} A \cos(\omega t_1 + \phi) A \cos(\omega t_0 + \phi) \frac{d\phi}{2\pi} \\ &= \frac{A^2}{2\pi} \cdot \frac{1}{2} \int_{-\pi}^{\pi} \cos(\omega(t_1 - t_0)) d\phi + \frac{A^2}{2\pi} \cdot \frac{1}{2} \int_{-\pi}^{\pi} \cos(\omega(t_1 + t_0) + 2\phi) d\phi \\ &= \frac{A^2}{2} \cos(\omega(t_1 - t_0)), \text{ function of } t_1 - t_0 \end{aligned}$$

For $t_1 - t_0 = \tau$,

$$R_X(t_1, t_0) = \frac{A^2}{2} \cos(\omega\tau) = R_X(\tau), \text{ } \tau \text{ is called the lag}$$

Example: $X[n]$ is a discrete-time random process with:

$$m[n] = 3, \quad R_X[n_1, n_0] = 9 + 4e^{-0.2|n_1 - n_0|}$$

Find the mean, variance, and covariance of $X_1 = X[8]$ and $X_0 = X[5]$.

Mean: $m_0 = m[5] = 3 \quad m_1 = m[8] = 3$

Variance: $E[X_0^2] = R_X[5,5] = 13, \quad E[X_1^2] = R_X[8,8] = 13$

$$\sigma_1^2 = \sigma_0^2 = 13 - 3^2 = 4$$

Covariance:

$$\text{cov}(X_1, X_0) = C_X[8,5] = R_X[8,5] - m[8]m[5] = 11.195 - 3 \cdot 3 = 2.195$$

Correlation coefficient:

$$\rho = \frac{\text{cov}(X_1, X_0)}{\sigma_1 \sigma_0} = \frac{2.195}{2 \cdot 2} = 0.5488$$

Wide-Sense Stationary Random Process

Continuous Time

A random process is said to be w.s.s. if and only if

$$(1) \quad m_X(t) = m = \text{constant}, \quad \text{for all } t$$

$$(2) \quad R_X(t_1, t_0) = R_X(t_1 - t_0) = R_X(\tau), \quad \text{for all } t_1, t_0 \quad \text{and} \quad \tau = t_1 - t_0$$

$$\text{or} \quad C_X(t_1, t_0) = C_X(t_1 - t_0) = C_X(\tau), \quad \text{for all } t_1, t_0 \quad \text{and} \quad \tau = t_1 - t_0$$

Discrete Time

Similar definitions apply.

Example:

- $X(t) = A \cos(\omega t + \phi)$, ϕ is uniform $[-\pi, \pi]$

$$R_X(t_1, t_0) = C_X(t_1, t_0) = \frac{A^2}{2} \cos(\omega(t_1 - t_0)) \quad \text{wss}$$

- Random telegraph signal

$$m_X(t) = (2p - 1)e^{-2\alpha t}$$

$$R_X(t_1, t_0) = e^{-2\alpha|t_1 - t_0|}$$

$$C_X(t_1, t_0) = e^{-2\alpha|t_1 - t_0|} - (2p - 1)^2 e^{-2\alpha|t_1 + t_0|} \quad \text{not wss}$$

Ergodicity

Let $X(t)$ be a wss process with $m_X(t) = m$, then it is said to be

- Mean ergodic if it satisfies

$$\lim_{T \rightarrow \infty} \langle X(t) \rangle_T = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(t) dt \doteq E[X(t)] = m$$

(Convergence in the mean-square sense.)

- Correlation ergodic if it satisfies

$$\begin{aligned} \lim_{T \rightarrow \infty} \langle X(t) X(t - \tau) \rangle_T &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(t) X(t - \tau) dt \\ &= E[X(t) X(t - \tau)] = R_X(\tau) \end{aligned}$$

Time Averages \equiv Ensemble Averages

Example: $X(t) = A$, where $A = \pm 1$ with equal probability:

$$m_X(t) = E[X(t)] = E[A] = 0;$$

$$\langle X(t) \rangle_T = \frac{1}{2T} \int_{-T}^T A dt = \frac{A}{2T} \int_{-T}^T dt = A$$

$$m_X(t) \neq \langle X(t) \rangle_T \quad \text{Therefore, not mean ergodic.}$$

Example: $X(t) = A \cos(\omega t + \phi)$ where ϕ is uniform $[-\pi, \pi]$

Ensemble average: $m_x(t) = E[X(t)] = AE[\cos(\omega t + \phi)] = 0$

Time average:

$$\begin{aligned} \langle X(t) \rangle_T &= \frac{1}{2T} \int_{-T}^T x(t) dt = \frac{A}{2T} \int_{-T}^T \cos(\omega t + \phi) dt = \frac{A}{2T} \left. \frac{\sin(\omega t + \phi)}{\omega} \right|_{-T}^T \\ &= \frac{A}{\omega 2T} [\sin(\omega T + \phi) - \sin(-\omega T + \phi)] \\ &= \frac{2A}{\omega 2T} \sin \omega T \cos \phi; \quad \lim_{T \rightarrow \infty} \frac{A}{\omega T} \sin \omega T \cos \phi = 0 \end{aligned}$$

Time average = Ensemble average
therefore mean ergodic

Example, cont'd.

Ensemble average: $R_X(\tau) = \frac{A^2}{2} \cos \omega \tau$

Time average:

$$\begin{aligned}\langle X(t)X(t-\tau) \rangle_T &= \frac{A^2}{2T} \int_{-T}^T \cos(\omega t + \phi) \cos(\omega(t-\tau) + \phi) dt \\ &= \frac{A^2}{4T} \int_{-T}^T \cos \omega \tau dt + \frac{A^2}{4T} \int_{-T}^T \cos(\omega(2t-\tau) + 2\phi) dt \\ &= \frac{A^2}{2} \cos \omega \tau - \frac{A^2}{8\omega T} \left[\sin(\omega(2t-\tau) + 2\phi) \right]_{-T}^T\end{aligned}$$

$$\lim_{T \rightarrow \infty} \langle X(t_1)X(t_0) \rangle_T = \frac{A^2}{2} \cos \omega \tau$$

Time averages = Ensemble averages
Mean and correlation ergodic

White Noise Process (*Continuous Time*)

Consider a zero-mean random process $X(t)$ with samples $X(t_0), X(t_1)$ taken at any arbitrary times t_0 and t_1 .

If these samples remain uncorrelated for any t_0 and t_1 , no matter how close, then this process must have infinite variance.

The correlation and covariance functions are of the form

$$R_X(t_1, t_0) = C_X(t_1, t_0) = \frac{N_0}{2} \delta(t_1 - t_0)$$

or

$$R_X(\tau) = C_X(\tau) = \frac{N_0}{2} \delta(\tau)$$

Such a process is known as *white noise*.

Example:

$$X(t) = A \cos(\omega t + \phi) + W(t),$$

- $W(t)$: white noise (mean is 0), power density $N_0/2$
- ϕ : uniform $[-\pi, \pi]$; ϕ, W are independent

$$\begin{aligned} R_X(t_1, t_0) &= E[X(t_1)X(t_0)] \\ &= E[\{A \cos(\omega t_1 + \phi) + W(t_1)\}\{A \cos(\omega t_0 + \phi) + W(t_0)\}] \\ &= A^2 E[\cos(\omega t_1 + \phi)\cos(\omega t_0 + \phi)] + E[w(t_1)W(t_0)] \\ &\quad + AE[\cos(\omega t_1 + \phi)W(t_0)] + AE[W(t_1)\cos(\omega t_0 + \phi)] \end{aligned}$$

Since W, ϕ are independent:

$$E[\cos(\omega t_1 + \phi)W(t_0)] = E[\cos(\omega t_1 + \phi)]E[W(t_0)] = 0$$

Example, cont'd.

$$\begin{aligned} R_x(t_1, t_0) &= A^2 E[\cos(\omega t_1 + \phi) \cos(\omega t_0 + \phi)] + E[W(t_1)W(t_0)] \\ &= \frac{A^2}{2} \cos \omega(t_1 - t_0) + \frac{A^2}{2} E[\cos(\omega(t_1 + t_0) + 2\phi)] + \frac{N_0}{2} \delta(t_1 - t_0) \\ &= \frac{A^2}{2} \cos \omega(t_1 - t_0) + \frac{N_0}{2} \delta(t_1 - t_0) \end{aligned}$$

or

$$R_x(\tau) = \frac{A^2}{2} \cos \omega \tau + \frac{N_0}{2} \delta(\tau)$$

Properties of the Autocorrelation Function†

1. $R_X(\tau) = R_X(-\tau)$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(t_1) R_X(t_1 - t_0) g(t_0) dt_1 dt_0 \geq 0$ for any function $g(t)$

(Property 2 is called the *positive semi-definite* property.)

Further properties of $R_X(\tau)$ that follow from the above:

$$R_X(0) = E[X^2(t)] \geq 0$$

$$|R_X(\tau)| \leq R_X(0)$$

† Apply also to the autocovariance function.

Cross-Correlation Function

$$R_{XY}(t_1, t_0) = E[X(t_1)Y(t_0)]$$

$X(t)$ and $Y(t)$ are jointly wide-sense stationary if $X(t)$ and $Y(t)$ are *each* wide-sense stationary and if

$$R_{XY}(t_1, t_0) = R_{XY}(t_1 - t_0) = R_{XY}(\tau)$$

Note that $R_{XY}(t_1, t_0) = E[Y(t_0)X(t_1)] = R_{YX}(t_0, t_1)$

\therefore if stationary: $R_{XY}(\tau) = R_{YX}(-\tau)$

Example: (*use of cross-correlation*)

$$X(t) = W(t) \quad \text{or} \quad X(t) = A \cos(\omega t + \phi) + W(t)$$

$$Y(t) = A \cos(\omega t + \phi) \quad (\phi \text{ is uniform } [-\pi, \pi])$$

Case (i): Let $X(t) = W(t)$, (signal absent)

$$\begin{aligned} R_{XY}(t_1, t_0) &= E[X(t_1)Y(t_0)] = AE[W(t_1)\cos(\omega t_0 + \phi)] \\ &= AE[W(t_1)]E[\cos(\omega t_0 + \phi)] = 0 \end{aligned}$$

Case (ii): Let $X(t) = A \cos(\omega t + \phi) + W(t)$, (signal present)

$$\begin{aligned} R_{XY}(t_1, t_0) &= E[X(t_1)Y(t_0)] = E[\{A \cos(\omega t_1 + \phi) + W(t_1)\}\{A \cos(\omega t_0 + \phi)\}] \\ &= A^2 E[\cos(\omega t_1 + \phi)\cos(\omega t_0 + \phi)] + AE[W(t_1)\cos(\omega t_0 + \phi)] \\ &= \frac{A^2}{2} \cos \omega(t_1 - t_0) = \frac{A^2}{2} \cos \omega \tau \end{aligned}$$

Power Spectral Density (PSD) Function

- Fourier transform of the autocorrelation function

$$S_X(f) = \mathcal{F}[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} df$$

- Properties:

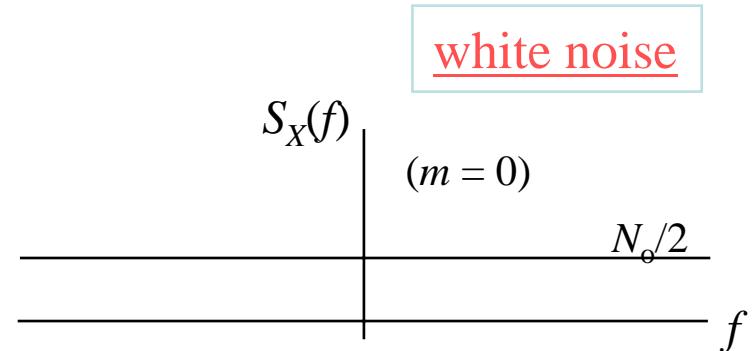
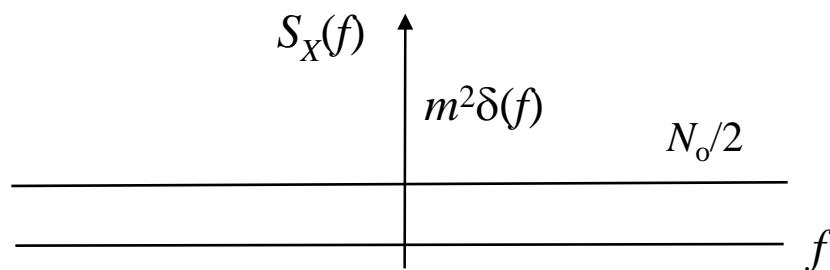
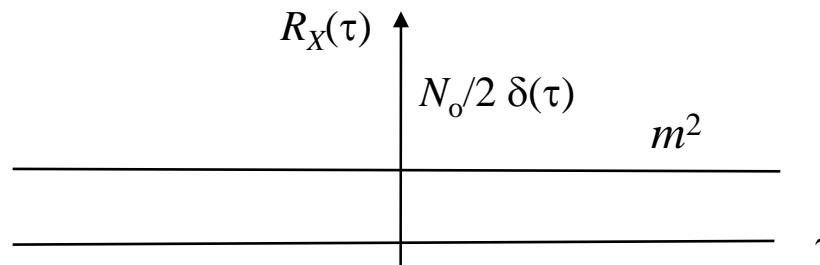
- $R_X(0) = E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f) df$
- $S_X(f)$ is real and even symmetric: $S_X(f) = S_X(-f)$
- $S_X(f) \geq 0$

Example:

- For a general uncorrelated random process

$$R_X(\tau) = C_X(\tau) + m^2 = \frac{N_0}{2} \delta(\tau) + m^2$$

$$S_X(f) = \mathcal{F}\left[\frac{N_0}{2} \delta(\tau) + m^2\right] = \frac{N_0}{2} + m^2 \delta(f)$$



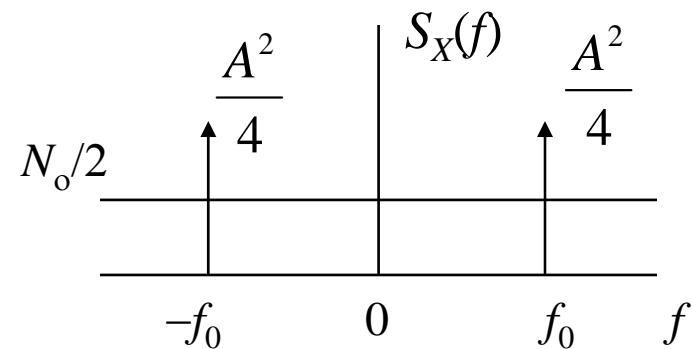
Example: $X(t) = A \cos(2\pi f_0 t + \phi) + W(t)$

$W(t)$ is white, $N_0/2$; ϕ is uniform $[-\pi, \pi]$; W, ϕ independent

$$R_X(\tau) = \frac{A^2}{2} \cos 2\pi f_0 \tau + \frac{N_0}{2} \delta(\tau)$$

$$S_X(f) = \mathcal{F}[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau} d\tau$$

$$= \frac{A^2}{2} \mathcal{F}[\cos 2\pi f_0 \tau] + \frac{N_0}{2} \mathcal{F}[\delta(\tau)] = \frac{A^2}{4} \delta(f - f_0) + \frac{A^2}{4} \delta(f + f_0) + \frac{N_0}{2}$$



Cross-Power Spectral Density Function

$$S_{XY}(f) = \mathcal{F}[R_{XY}(\tau)]$$

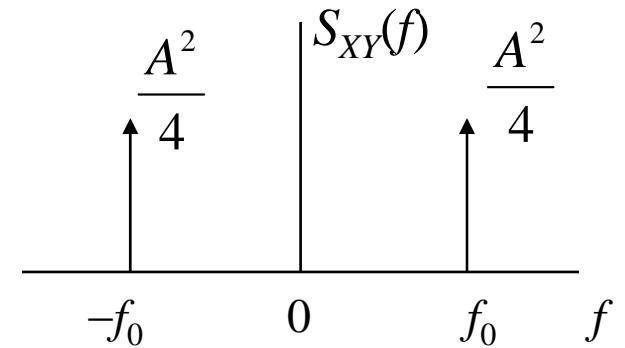
Example: $X(t) = A \cos(2\pi f_0 t + \phi) + W(t)$, $Y(t) = A \cos(2\pi f_0 t)$

$W(t)$ is white, $N_0/2$; ϕ is uniform $[-\pi, \pi]$; W, ϕ independent

$$R_{XY}(\tau) = \frac{A^2}{2} \cos 2\pi f_0 \tau$$

$$S_{XY}(\omega) = \mathcal{F}[R_{XY}(\tau)] = \frac{A^2}{2} \mathcal{F}[\cos 2\pi f_0 \tau]$$

$$= \frac{A^2}{2} \delta(f - f_0) + \frac{A^2}{4} \delta(f + f_0)$$



Simulation Example: $X(t) = 4 \cos(10\pi t + 0.59) + W(t)$

Noise $W(t)$ is white, $m = 0$, $N_0/2 = 0.25$ watts/Hz

Phase $\phi = 0.59$ radians (single realization)

Assumption: ϕ is uniform $[-\pi, \pi]$; W, ϕ are independent

$$SNR_{dB} = 10 \log_{10} (\text{Signal Power}/\text{Noise Power})$$

Waveform duration: $0 \leq t < 2$ seconds

Generate 200 samples with sampling period: $T_S = 0.01$ sec

Sampling frequency: $f_S = 100$ Hz

Noise Power = 0.25 watts/Hz $\times 100$ Hz = 25 watts

$$SNR_{dB} = 10 \log_{10} \left(\frac{4^2}{2/25} \right) = -4.95 \text{ dB}$$

Simulation Example (cont'd.)

Correlations and power spectrum:

MATLAB **xcorr** produces 399 samples

Taking the fft of the middle 200 values:

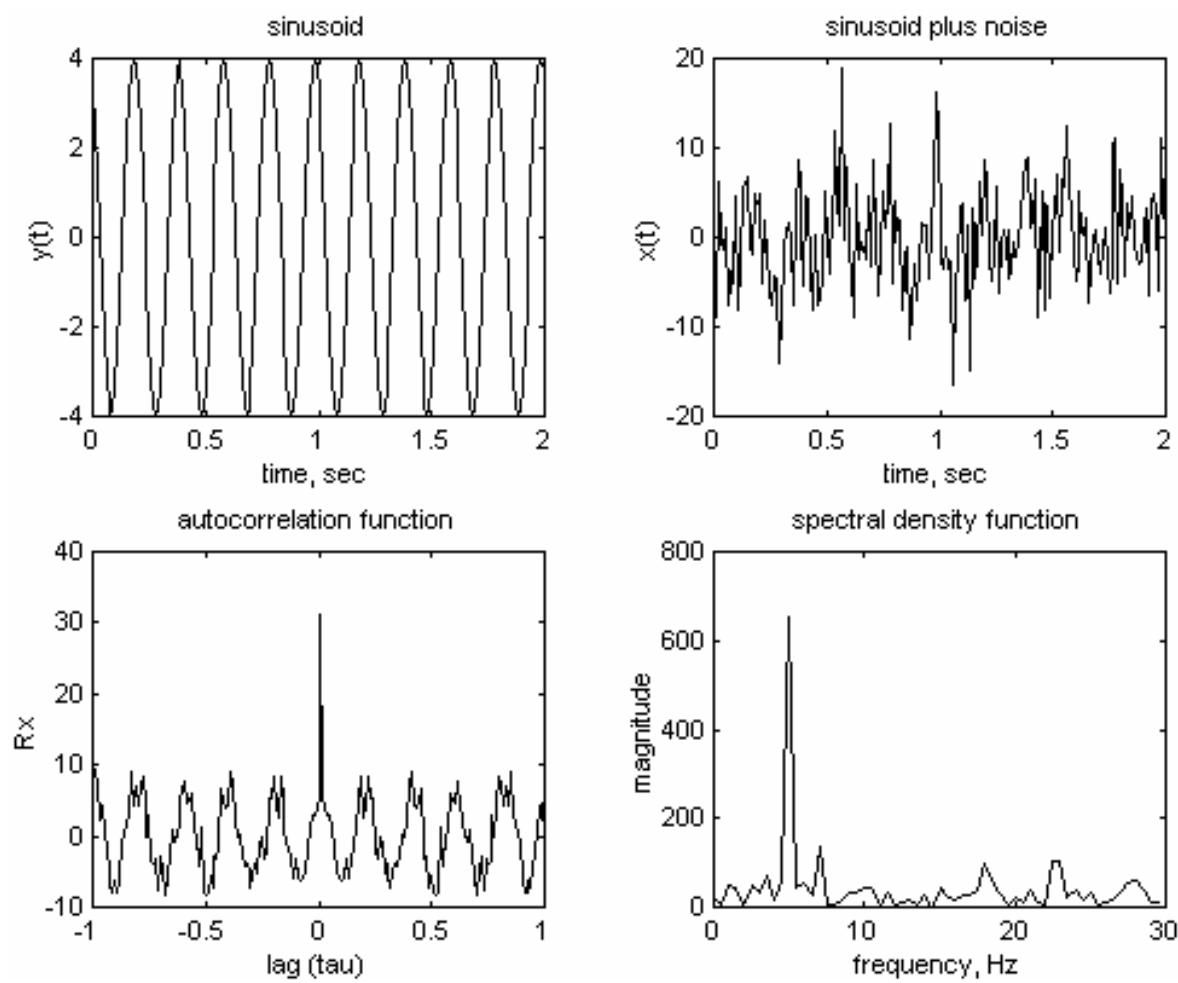
- 200 fft coefficients covering 0 to 100 Hz
- bin width is $100/200 = 0.5$ Hz
- plot the first 60 fft coefficients

Phase ϕ is uniform

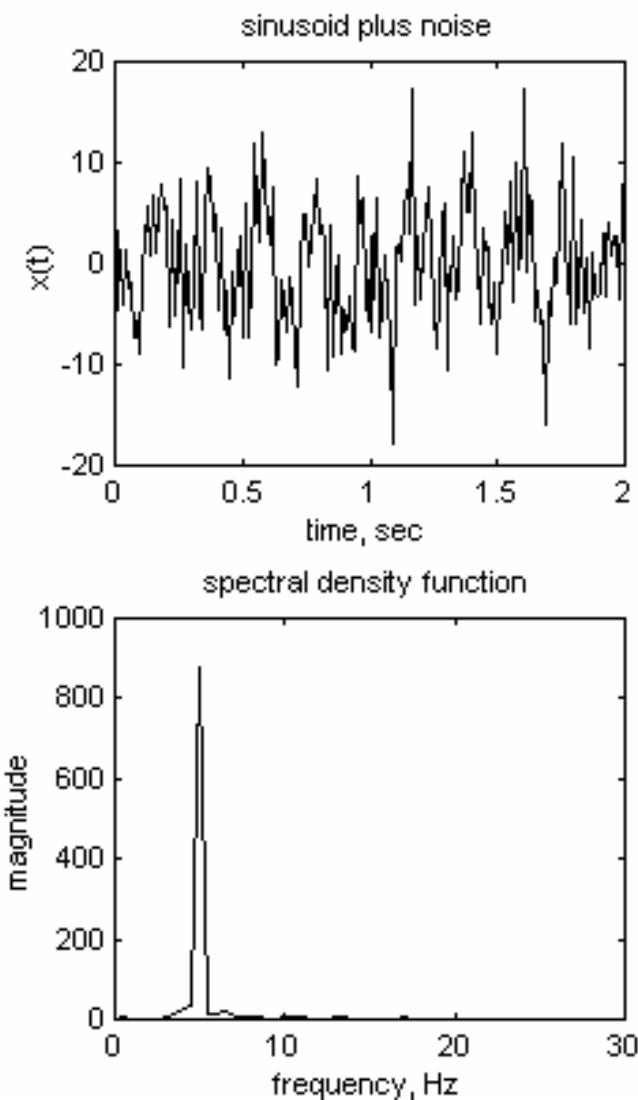
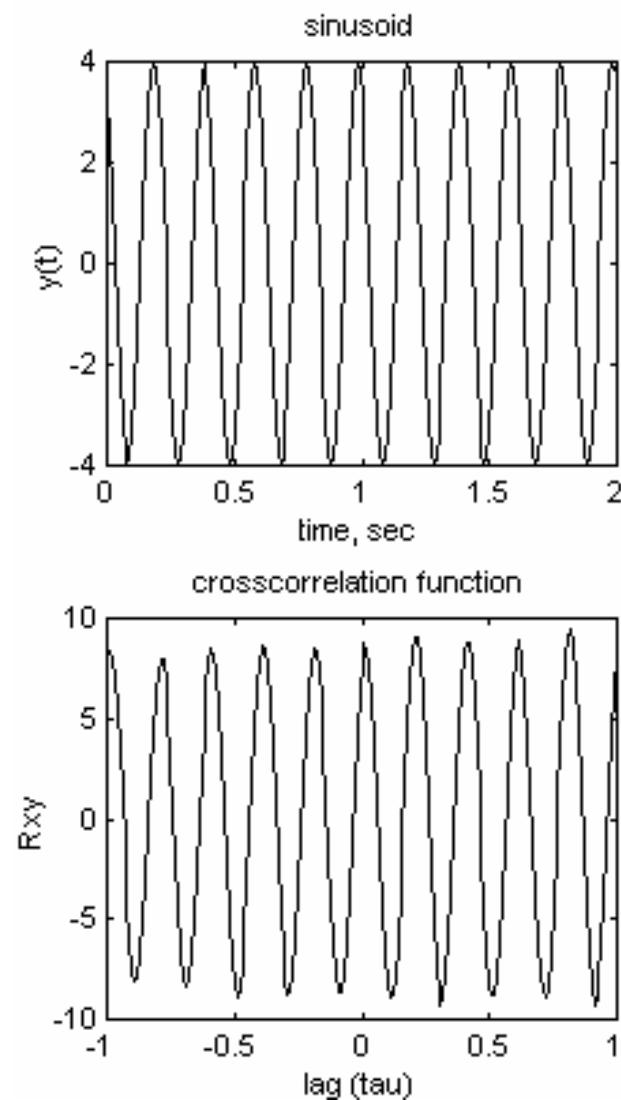
Run simulations for different realizations of ϕ : $[-\pi, \pi]$.

Average the results.

Autocorrelation and Spectral Density Function:



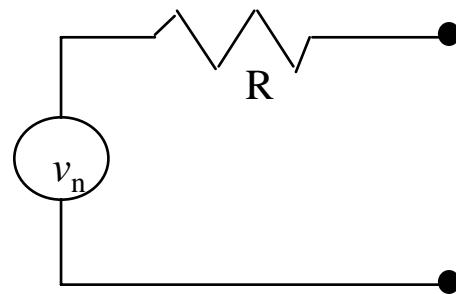
Cross-Correlation and Cross-Spectrum:



Thermal Noise

- Occurs due to randomness of the thermally excited motion of free electrons in a non-ideal conducting medium.
 - The free electron motion gives rise to a fluctuating voltage.
 - The total noise voltage is given by the sum of a very large number of (independently produced) voltage pulses of very short duration.
 - By applying the central limit theorem, the total noise voltage would be a white Gaussian process.

- Thermal noise can be represented as a combination of an ideal resistor and a noise generator:



- The power spectral density of the equivalent noise generator is

$$S_w(f) = \frac{kT}{2} = \frac{N_0}{2} \text{ watts/Hz}$$

where T is the temperature of the medium in °Kelvin and $k = 1.37 \times 10^{-23}$ Joules/ °Kelvin is Boltzmann's constant.

Example:

The power spectral density due to thermal noise in a $1 \text{ M}\Omega$ resistor at 25°C is computed as follows:

$$T = 25 + 273.16 = 298.16 \quad (\text{in degrees Kelvin})$$

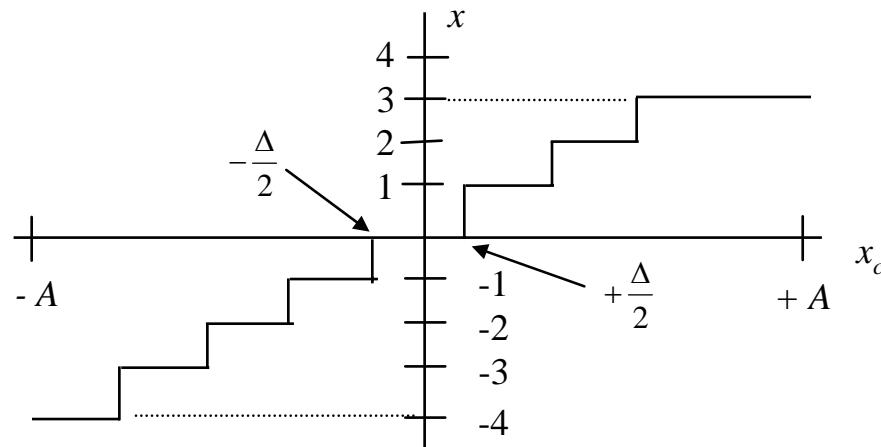
$$S_w(f) = \frac{kT}{2} = \frac{N_0}{2} = \frac{1.38 \times 10^{-23} \times 298.16}{2} = 2.0573 \times 10^{-21} \text{ watts/Hz}$$

If the bandwidth of the system is 10 MHz, the total noise power contributed by the resistor is 2.0573×10^{-14} watts.

Quantization Noise

- Quantization noise occurs whenever a continuous magnitude value is discretized.
- Quantization is achieved via rounding or truncation.
- Quantization step size:

$$\Delta = 2A/N$$



Let $N = 2^b$ be the number of quantization levels; b the number of bits.

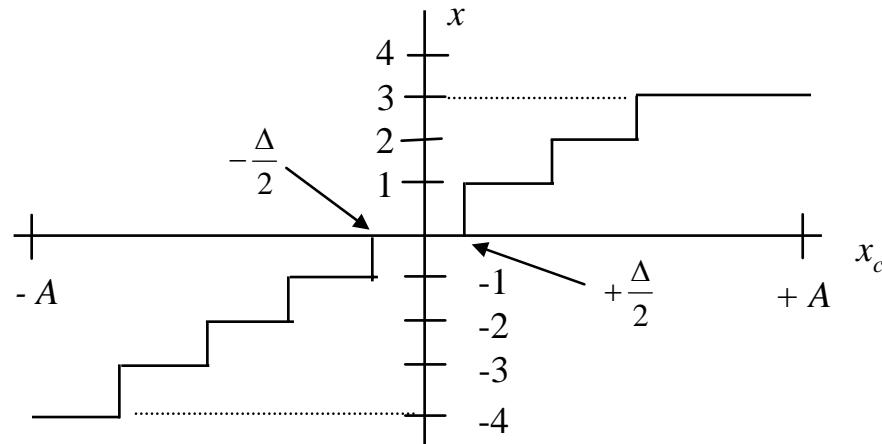
Quantization error: $e = x_c - x$, which is in the range: $\pm \Delta/2 = \pm A/N$.

Because of rounding, e is uniformly distributed: $[-\Delta/2, \Delta/2]$

$$f_E(e) = \frac{1}{\Delta}$$

$$-\frac{\Delta}{2} \leq e \leq \frac{\Delta}{2}$$

The mean of e is zero;
variance: $\Delta^2/12 = 4A^2/12N^2$.

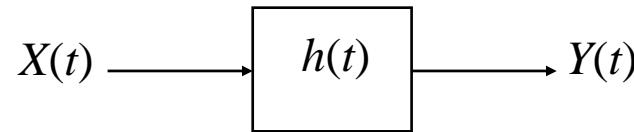


Assuming sinusoidal input, the SNR is then given by:

$$\begin{aligned} SNR_{dB} &= 10 \log_{10} \left(\frac{\text{Signal Power}}{\text{Noise Power}} \right) = 10 \log_{10} \left(\frac{A^2/2}{4A^2/12N^2} \right) \\ &= 10 \log_{10} \left(1.5 \times 2^{2b} \right) = 1.7609 + 6.0206b \text{ dB} \end{aligned}$$

Response of Linear Systems to Random Signals

w.s.s. LTI System



$$Y(t) = \int_{-\infty}^{\infty} h(u) X(t-u) du = h(t)^* X(t)$$

Mean of the system output:

$$\begin{aligned} E[Y(t)] &= E\left[\int_{-\infty}^{\infty} h(u) X(t-u) du\right] = \int_{-\infty}^{\infty} h(u) E[X(t-u)] du \\ &= \int_{-\infty}^{\infty} h(u) m_x du = m_x \int_{-\infty}^{\infty} h(u) du \end{aligned}$$

$$m_y = m_x \int_{-\infty}^{\infty} h(u) du$$

Cross-correlation function:

$$\begin{aligned} R_{YX}(\tau) &= E[Y(t)X(t-\tau)] = E\left[\left(\int_{-\infty}^{\infty} h(u)X(t-u)du\right)X(t-\tau)\right] \\ &= \int_{-\infty}^{\infty} h(u)E[X(t-u)X(t-\tau)]du = \int_{-\infty}^{\infty} h(u)R_X(\tau-u)du \end{aligned}$$

$$R_{YX}(\tau) = h(\tau)^* R_X(\tau)$$

Output autocorrelation function:

$$\begin{aligned} R_Y(\tau) &= E[Y(t)Y(t-\tau)] = E\left[\left(\int_{-\infty}^{\infty} h(u)X(t-u)du\right)Y(t-\tau)\right] \\ &= \int_{-\infty}^{\infty} h(u)E[X(t-u)Y(t-\tau)]du = \int_{-\infty}^{\infty} h(u)R_{XY}(\tau-u)du \end{aligned}$$

then:

$$\begin{aligned} R_Y(\tau) &= h(\tau)^* R_{XY}(\tau) \\ &= h(\tau)^* R_{YX}(-\tau) = h(\tau)^* \{h(-\tau)^* R_X(-\tau)\} \\ &= h(\tau)^* h(-\tau)^* R_X(-\tau) \end{aligned}$$

$$R_Y(\tau) = h(\tau)^* h(-\tau)^* R_X(\tau)$$

Frequency Domain Analysis

$$\begin{array}{ll} R_X(\tau) \Leftrightarrow S_X(f) & R_Y(\tau) \Leftrightarrow S_Y(f) \\ R_{XY}(\tau) \Leftrightarrow S_{XY}(f) & R_{YX}(\tau) \Leftrightarrow S_{YX}(f) \\ h(\tau) \Leftrightarrow H(f) & h(-\tau) \Leftrightarrow H^*(f) \end{array}$$

Cross-spectral density function:

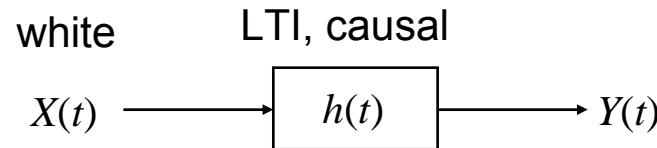
$$R_{YX}(\tau) = h(\tau)^* R_X(\tau) \Leftrightarrow S_{YX}(f) = H(f) S_X(f)$$

Output power spectral density function

$$R_Y(\tau) = h(\tau)^* h(-\tau)^* R_X(\tau) \Leftrightarrow S_Y(f) = |H(f)|^2 S_X(f)$$

Application: System Identification

- The system impulse response $h(t)$ is not known.



- Let the system input be white noise: $R_X(\tau) = \delta(\tau)$; $S_X(f) = 1$
- The system response is identified by computing cross-correlation between input and output.

$$\text{Time (lag) domain: } R_{YX}(\tau) = h(\tau)^* R_X(\tau) = h(\tau)^* \delta(\tau) = h(\tau)$$

$$\text{Frequency domain: } S_{YX}(f) = H(f) S_X(f) = H(f) \cdot 1 = H(f)$$

Simulation Example:

